Why study moduli problems? Two reasons...

(A) Classify geometric objects

leads to deeper understanding of the structure of these objects

Ex: compact Lie groups $\rightsquigarrow$ theory of root data

(B) Explosion of interest in moduli since 80's

New invariants in differential geometry:

pseudoholomorphic curves, gauge theory

Idea: Given a manifold X...

Associate a moduli problem $\mathcal{M}(X)$, then "count" $\#\mathcal{M}(X)$, objects up to isomorphism.

When X is a projective variety, M(X) often has analogs in algebraic geometry

easier, thanks to language of stacks

First issue:

some objects are parameterized by continuous parameters

Ex: moduli of curves as an $\underbrace{\text{algebraic stack}}_{=\text{functor of points}}$

For any scheme B,

$$\mathcal{M}_g(B) := \underbrace{\left\{ \begin{array}{l} \text{smooth families of} \\ \text{genus } g \text{ curves} / B \end{array} \right\}}_{\text{groupoid}}$$

An algebraic stack which is "Deligne-Mumford" (DM)

$\Longrightarrow$ Finite automorphism groups

Gold standard

Thm (Keel-Mori): Any separated DM stack $\mathcal{M}$ has a coarse moduli space $\mathcal{M} \to X$

Ignore: space vs. scheme vs. quasi-projective scheme

This suggests a general approach:

1) Identify a functor of points
2) Show that it is an algebraic stack
   ↳ e.g. Artin's criteria
3) Check that it is separated and DM, and apply Keel-Mori theorem

Problem (for either goals A or B): Many stacks of interest are not DM, so (3) fails

Today: A version of this program which works for general algebraic stacks

Gold standard:

Fix smooth genus g curve, C

$$\mathrm{Bun}_{r,d}(B) := \left\{ \begin{array}{l} \text{vector bundles on } C \times B \text{ with} \\ \deg = d, \ \mathrm{rank} = r \text{ on fibers} \end{array} \right\}$$

This is an algebraic stack

⟶ can discuss line bundles, sheaves, cohomology, open/closed substacks

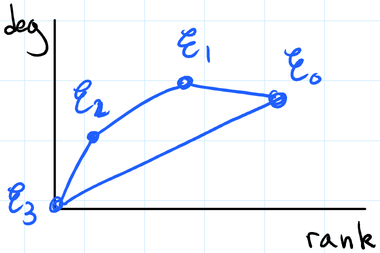
Ex: a line bundle is a natural assignment

$$\begin{pmatrix} \text{maps} \\ B \to \mathrm{Bun}_{r,d} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{line bundles} \\ \text{on } B \end{pmatrix}$$

But doesn't help much with goal A, classification:

$\mathrm{Bun}_{r,d}$ is unbounded, non-separated (look at degenerations of $\mathcal{O}_C(-n) \oplus \mathcal{O}_C(n)$)

Thm (Harder-Narasimhan): Every bundle has a unique filtration $\mathcal{E} = \mathcal{E}_0 \supsetneq \mathcal{E}_1 \supsetneq \cdots \supsetneq \mathcal{E}_p = 0$ such that

1) 

slope$(\mathcal{E}_i/\mathcal{E}_{i+1})$ increasing

↖ HN polytope

2) $\mathcal{E}_i/\mathcal{E}_{i+1}$ has no sub-bundle of larger slope (i.e. semistable)

Thm (Shatz, Mumford): there exists a stratification by locally closed substacks

$$\mathrm{Bun}_{r,d} = \mathrm{Bun}_{r,d}^{ss} \cup \bigcup_{\alpha} S_\alpha \quad \rightsquigarrow \quad \text{bundles of HN type } \alpha = \begin{pmatrix} r_1 & \cdots & r_p \\ d_1 & \cdots & d_p \end{pmatrix}$$

$$S_\alpha \underset{\oplus}{\overset{gr}{\rightleftarrows}} Z_\alpha^{ss} = \mathrm{Bun}_{r_1,d_1}^{ss} \times \cdots \times \mathrm{Bun}_{r_p,d_p}^{ss}$$

Also, $\mathrm{Bun}_{r,d}$ represented by smooth projective scheme if r & d coprime (has "good moduli space" in general).

More interesting: sheaves on a surface S

$$\mathfrak{X}_v = \left\{ \begin{array}{l} \text{coherent sheaves on } S \text{ with} \\ \text{numerical K-theory class } v \end{array} \right\}$$

Exact same structure, but HN stratification depends on an ample class $H \in NS(S)_{\mathbb{R}}$:

$$\mathfrak{X}_v = \mathfrak{X}_v^{H\text{-}ss} \cup \bigcup_\alpha S_\alpha \quad \rightsquigarrow \quad \alpha = \left\{ \begin{array}{l} (v_1, \ldots, v_p) \in K_0^{num}(S) \\ v_1 + \cdots + v_p = v \end{array} \right\}$$

$$\downarrow$$

$M_v^{H\text{-}ss}$ moduli space

As H varies... M_v^H changes $\rightsquigarrow$ wall & chamber structure on $NS(S)_{\mathbb{R}}$

Donaldson invariants of S arise as "integrals"

$$\text{“}\int_{M_v^{H\text{-}ss}}\text{”} \left(\begin{array}{l} \text{tautological} \\ \text{cohomology class} \end{array} \right)$$

Q: How do they depend on H?

Ex: Bridgeland semistable objects in $\mathcal{A} \subset D^b(S)$

$\longrightarrow$ New techniques needed

Good moduli spaces:

$\mathfrak{X}$ is a finite type stack, affine diagonal

> Def: a good moduli space (GMS) is a map to a space $q: \mathfrak{X} \to M$ s.t.
>
> i) $q_*: QCoh(\mathfrak{X}) \to QCoh(M)$ is exact
>
> ii) $q_*(\mathcal{O}_{\mathfrak{X}}) = M$

Properties:

1) Fibers of q = "S-equivalence" classes
2) Universal for maps to spaces (categorical quotient)
3) Example: $X^{ss}/G \to X^{ss}/\!\!/G$, reductive GIT
4) If M is proper, dim $H^*(\mathfrak{X}, E) < \infty$
5) Alper-Hall-Rydh '16: étale locally over M, looks like $Spec(A)/G \to Spec(A^G)$

Stability

$\mathfrak{X}$ = stack of coherent sheaves on S

Rees correspondence:

> ($\mathbb{Z}$-weighted filtrations $\cdots \supset \mathcal{E}_w \supset \mathcal{E}_{w+1} \supset \cdots$)
>
> $\Updownarrow$
>
> ($\mathbb{C}^*$-equiv. coherent sheaves on $S \times \mathbb{C}^1$, flat over S)
>
> $\Updownarrow$
>
> (maps of stacks $\Theta := \mathbb{C}^1/\mathbb{C}^* \to \mathfrak{X}$)

Gives an intrinsic formulation of the HN filtration:

a canonical map $f: \Theta \to \mathfrak{X}$

How to find it? Use numerical invariant.

Given $f: \Theta \to \mathfrak{X}$, define: (assuming $\deg(v) = 0$)

$$\mu(f) = \frac{-\sum w \deg(\mathcal{E}_w/\mathcal{E}_{w+1})}{\sqrt{\sum w^2 \operatorname{rank}(\mathcal{E}_w/\mathcal{E}_{w+1})}}$$

Thm (HL, Hoskins, Zamora): Among all maps $f : \Theta \to \mathcal{X}$, there one which maximizes $\mu(f)$, unique up to ramified cover $\Theta \xrightarrow{z^n} \Theta$.

Now let $\mathcal{X}$ be arbitrary, and fix $\ell \in H^2(\mathcal{X})$ and $b \in H^4(\mathcal{X})$.

1) $p \in \mathcal{X}$ is semistable if $f^*(\ell) \leq 0$ in $H^2(\Theta) \cong \mathbb{Z}$ for all filtrations of p

2) [HN problem] For unstable $p \in \mathcal{X}$, find f which maximizes
$$\mu(f) = \frac{f^* \ell}{\sqrt{f^* b}}$$

Might lead to a Θ-stratification:

$$\mathcal{X} = \mathcal{X}^{ss} \cup \bigcup S_\alpha \text{, where}$$

$$\begin{array}{lll} S_\alpha & \rightsquigarrow & \text{parameterize filtered objects } \Theta \to \mathcal{X} \\ \text{gr} \downarrow \uparrow & & \\ Z_\alpha^{ss} & \rightsquigarrow & \text{parameterize graded objects } */\mathbb{C}^* \to \mathcal{X} \end{array}$$

Ideal solution to moduli problem:

Θ-stratification where $\mathcal{X}^{ss}$ and Z_α^{ss} have GMS

Main theorem of GIT:

Let $q : \mathcal{X} \to Y$ be a good moduli space, and fix a form $b \in H^4(\mathcal{X})$.

Thm: $\forall \ell \in NS(\mathcal{X})_{\mathbb{R}}$, numerical invariant $\mu = \ell/\sqrt{b}$ defines a Θ-stratification

$$\mathcal{X} = \mathcal{X}^{ss}(\ell) \cup S_0 \cup \cdots \cup S_N \quad \text{and} \quad S_i \rightleftarrows Z_i^{ss},$$

where $\mathcal{X}^{ss}(\ell)$ and Z_i^{ss} have GMS which are projective $/Y$

Also includes variation of GIT quotient:

if $\mathcal{X}$ is irreducible, as ℓ varies get birational modifications

$$\begin{tikzcd} \mathcal{X}^{ss}(\ell) \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{X} \arrow[dd] & \mathcal{X}^{ss}(\ell') \arrow[l, phantom, "\supset"] \arrow[d] \\ M^{ss}(\ell) \arrow[rr, dashed] \arrow[dr] & & M^{ss}(\ell') \arrow[dl] \\ & Y & \end{tikzcd}$$

Meta-principal:
Birational geometry of moduli spaces should be understood as variation of stability in some larger moduli problem with good moduli space

$$q: \mathfrak{X} \longrightarrow Y$$

Ex: Smyth classified all DM modular compactifications of $\overline{M}_{g,n}$

↳ Would be nice to run MMP on $\overline{M}_{g,n}$ by varying stability on moduli of all curves

Ex: Bayer-Macri prove that if

$\underset{\text{projective CY}}{X} \dashrightarrow M_v^{H\text{-ss}}(\underset{K3}{S})$, then $X \cong \left\{ \begin{array}{l} \text{Bridgeland semistable} \\ \text{complexes on some} \\ \text{twisted K3} \end{array} \right\}$

We will use this in the next lecture to study the local structure of flops

Useful concept for constructing Θ-stratifications

> Def: $\mathfrak{X}$ is Θ-reductive if for any family over a discrete valuation ring, $\mathrm{Spec}(R) \to \mathfrak{X}$, any filtration over the generic fiber extends uniquely to the special fiber.

> Thm A (HL): Let $\mathfrak{X}$ be a Θ-reductive algebraic stack. Then a numerical invariant μ defines a Θ-stratification if and only if
>
> 1) Every unstable point has a unique HN filtration
>
> 2) In a bounded family $\mathrm{Spec}(A) \to \mathfrak{X}$, only finitely many types of HN filtrations arise

Rem: main theorem of GIT is a special case

The notion of Θ-reductive stack is useful for constructing good moduli spaces as well

Thm B (Alper-HL-Heinloth): Let $\mathcal{X}$ be locally finite type with affine diagonal. Then $\mathcal{X}$ has a good moduli space if and only if

1) $\mathcal{X}$ is Θ-reductive,
2) closed points of $\mathcal{X}$ have reductive automorphism groups, and
3) $\mathcal{X}$ has "unpunctured inertia"

These two theorems provide a program for analyzing general moduli problems, analogous to Keel-Mori theorem

Next time: Discuss what thm B means, and applications to Bridgeland semistable complexes